

Vectors and Functions in Space and Time

RA Gillmann, July 2020

The rate of an n -dimensional vector variable or vector-valued function $\Delta \mathbf{x}$ per unit of an independent scalar variable t equals

$$\frac{\Delta \mathbf{x}}{\Delta t} := \left(\frac{\Delta x_1}{\Delta t}, \frac{\Delta x_2}{\Delta t}, \dots, \frac{\Delta x_n}{\Delta t} \right).$$

The same except with Δt and Δx , respectively:

$$\frac{\Delta \mathbf{t}}{\Delta x} := \left(\frac{\Delta t_1}{\Delta x}, \frac{\Delta t_2}{\Delta x}, \dots, \frac{\Delta t_n}{\Delta x} \right).$$

The rate of a scalar variable Δx per unit of an independent n -dimensional vector variable or vector-valued function $\Delta \mathbf{t}$ equals

$$\frac{\Delta x}{\Delta \mathbf{t}} := \left(\frac{\Delta x}{\Delta t_1}, \frac{\Delta x}{\Delta t_2}, \dots, \frac{\Delta x}{\Delta t_n} \right).$$

The same except with Δt and $\Delta \mathbf{x}$, respectively:

$$\frac{\Delta t}{\Delta \mathbf{x}} := \left(\frac{\Delta t}{\Delta x_1}, \frac{\Delta t}{\Delta x_2}, \dots, \frac{\Delta t}{\Delta x_n} \right).$$

The rate of an n -dimensional vector variable or vector-valued function $\Delta \mathbf{x}$ per unit of an independent n -dimensional vector variable or vector-valued function $\Delta \mathbf{t}$ equals

$$\frac{\Delta \mathbf{x}}{\Delta \mathbf{t}} := \left(\frac{\Delta \mathbf{x}}{\Delta t_1}, \frac{\Delta \mathbf{x}}{\Delta t_2}, \dots, \frac{\Delta \mathbf{x}}{\Delta t_n} \right) = \begin{pmatrix} \frac{\Delta x_1}{\Delta t_1} & \dots & \frac{\Delta x_1}{\Delta t_n} \\ \vdots & \ddots & \vdots \\ \frac{\Delta x_n}{\Delta t_1} & \dots & \frac{\Delta x_n}{\Delta t_n} \end{pmatrix}.$$

The same except with Δt and $\Delta \mathbf{x}$, respectively:

$$\frac{\Delta \mathbf{t}}{\Delta \mathbf{x}} := \left(\frac{\Delta \mathbf{t}}{\Delta x_1}, \frac{\Delta \mathbf{t}}{\Delta x_2}, \dots, \frac{\Delta \mathbf{t}}{\Delta x_n} \right) = \begin{pmatrix} \frac{\Delta t_1}{\Delta x_1} & \dots & \frac{\Delta t_1}{\Delta x_n} \\ \vdots & \ddots & \vdots \\ \frac{\Delta t_n}{\Delta x_1} & \dots & \frac{\Delta t_n}{\Delta x_n} \end{pmatrix}.$$

The derivative of an n -dimensional vector function \mathbf{x} with respect to an independent scalar variable t equals

$$\frac{d\mathbf{x}(t)}{dt} := \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \dots, \frac{dx_n}{dt} \right)$$

The same except with \mathbf{t} and x , respectively:

$$\frac{d\mathbf{t}(x)}{dx} := \left(\frac{dt_1}{dx}, \frac{dt_2}{dx}, \dots, \frac{dt_n}{dx} \right)$$

The directional derivative of a scalar function x with respect to an independent n -dimensional vector variable \mathbf{t} equals

$$\frac{dx(\mathbf{t})}{d\mathbf{t}} := \left(\frac{\partial x}{\partial t_1}, \frac{\partial x}{\partial t_2}, \dots, \frac{\partial x}{\partial t_n} \right),$$

The same except with t and \mathbf{x} , respectively:

$$\frac{dt(\mathbf{x})}{d\mathbf{x}} := \left(\frac{\partial t}{\partial x_1}, \frac{\partial t}{\partial x_2}, \dots, \frac{\partial t}{\partial x_n} \right).$$

The directional derivative of an n -dimensional vector-valued function \mathbf{x} with respect to an n -dimensional vector variable \mathbf{t} equals

$$\frac{\partial \mathbf{x}(\mathbf{t})}{\partial \mathbf{t}} := \left(\frac{\partial \mathbf{x}}{\partial t_1}, \frac{\partial \mathbf{x}}{\partial t_2}, \dots, \frac{\partial \mathbf{x}}{\partial t_n} \right) = \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \dots & \frac{\partial x_1}{\partial t_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t_1} & \dots & \frac{\partial x_n}{\partial t_n} \end{pmatrix},$$

The same except with \mathbf{t} and \mathbf{x} , respectively:

$$\frac{\partial \mathbf{t}(\mathbf{x})}{\partial \mathbf{x}} := \left(\frac{\partial \mathbf{t}}{\partial x_1}, \frac{\partial \mathbf{t}}{\partial x_2}, \dots, \frac{\partial \mathbf{t}}{\partial x_n} \right) = \begin{pmatrix} \frac{\partial t_1}{\partial x_1} & \dots & \frac{\partial t_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial t_n}{\partial x_1} & \dots & \frac{\partial t_n}{\partial x_n} \end{pmatrix}.$$

The matrices are square Jacobian matrices, whose (i, j) th entry is

$$\mathbf{J}_{ij} = \frac{\partial x_i}{\partial t_j} \text{ or } \frac{\partial t_i}{\partial x_j}.$$

The inverse function theorem states that the matrix inverse of the Jacobian matrix of an invertible function is the Jacobian matrix of the inverse function.