Vectors and Functions in Space and Time

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The quotient of an *n*-dimensional vector variable or vector-valued function **x** with a scalar variable *t* equals

$$\frac{\mathbf{x}}{t} := \left(\frac{x_1}{t}, \frac{x_2}{t}, \dots, \frac{x_n}{t}\right).$$

The same except with **t** and *x*, respectively, is:

$$\frac{\mathbf{t}}{x} := \left(\frac{t_1}{x}, \frac{t_2}{x}, \dots, \frac{t_n}{x}\right).$$

The rate of an *n*-dimensional vector variable or vector-valued function Δ**x** per unit of an independent scalar variable *t* equals

$$\frac{\Delta\mathbf{x}}{\Delta t} := \left(\frac{\Delta x_1}{\Delta t}, \frac{\Delta x_2}{\Delta t}, \dots, \frac{\Delta x_n}{\Delta t}\right).$$

The same except with Δ**t** and Δ*x*, respectively, is:

$$\frac{\Delta\mathbf{t}}{\Delta x} := \left(\frac{\Delta t_1}{\Delta x}, \frac{\Delta t_2}{\Delta x}, \dots, \frac{\Delta t_n}{\Delta x}\right).$$

The rate of a scalar variable Δ*x* per unit of an independent *n*-dimensional vector variable or vector-valued function Δ**t** equals

$$\frac{\Delta x}{\Delta\mathbf{t}} := \left(\frac{\Delta x}{\Delta t_1}, \frac{\Delta x}{\Delta t_2}, \dots, \frac{\Delta x}{\Delta t_n}\right).$$

The same except with Δ*t* and Δ**x**, respectively, is:

$$\frac{\Delta t}{\Delta\mathbf{x}} := \left(\frac{\Delta t}{\Delta x_1}, \frac{\Delta t}{\Delta x_2}, \dots, \frac{\Delta t}{\Delta x_n}\right).$$

The rate of an *n*-dimensional vector variable or vector-valued function Δ**x** per unit of an independent *n*-dimensional vector variable or vector-valued function Δ**t** equals

$$\frac{\Delta\mathbf{x}}{\Delta\mathbf{t}} := \left(\frac{\Delta\mathbf{x}}{\Delta t_1}, \frac{\Delta\mathbf{x}}{\Delta t_2}, \dots, \frac{\Delta\mathbf{x}}{\Delta t_n}\right) = \begin{pmatrix} \frac{\Delta x_1}{\Delta t_1} & \cdots & \frac{\Delta x_1}{\Delta t_n} \\ \vdots & \ddots & \vdots \\ \frac{\Delta x_n}{\Delta t_1} & \cdots & \frac{\Delta x_n}{\Delta t_n} \end{pmatrix}.$$

The same except with Δ**t** and Δ**x**, respectively, is:

$$\frac{\Delta\mathbf{t}}{\Delta\mathbf{x}} := \left(\frac{\Delta\mathbf{t}}{\Delta x_1}, \frac{\Delta\mathbf{t}}{\Delta x_2}, \dots, \frac{\Delta\mathbf{t}}{\Delta x_n}\right) = \begin{pmatrix} \frac{\Delta t_1}{\Delta x_1} & \cdots & \frac{\Delta t_1}{\Delta x_n} \\ \vdots & \ddots & \vdots \\ \frac{\Delta t_n}{\Delta x_1} & \cdots & \frac{\Delta t_n}{\Delta x_n} \end{pmatrix}.$$

The derivative of an *n*-dimensional vector function **x** with respect to an independent scalar variable *t* equals

$$\frac{\mathbf{dx}(t)}{dt} \coloneqq \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \dots, \frac{dx_n}{dt}\right)$$

The same except with **t** and *x*, respectively, is:

$$\frac{\boldsymbol{d}\mathbf{t}(x)}{dx} \coloneqq \left(\frac{dt_1}{dx}, \frac{dt_2}{dx}, \dots, \frac{dt_n}{dx}\right)$$

The directional derivative of a scalar function *x* with respect to an independent *n*-dimensional vector variable **t** equals

$$\frac{dx(\mathbf{t})}{\mathbf{dt}} := \left(\frac{\partial x}{\partial t_1}, \frac{\partial x}{\partial t_2}, \dots, \frac{\partial x}{\partial t_n}\right),$$

The same except with *t* and **x**, respectively, is:

$$\frac{dt(\mathbf{x})}{\mathbf{dx}} := \left(\frac{\partial t}{\partial x_1}, \frac{\partial t}{\partial x_2}, \dots, \frac{\partial t}{\partial x_n}\right).$$

The directional derivative of an *n*-dimensional vector-valued function **x** with respect to an *n*-dimensional vector variable **t** equals

$$\frac{\partial \mathbf{x}(\mathbf{t})}{\partial \mathbf{t}} := \left(\frac{\partial \mathbf{x}}{\partial t_1}, \frac{\partial \mathbf{x}}{\partial t_2}, \dots, \frac{\partial \mathbf{x}}{\partial t_n}\right) = \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_n} \end{pmatrix},$$

The same except with **t** and **x**, respectively, is:

$$\frac{\partial \mathbf{t}(\mathbf{x})}{\partial \mathbf{x}} := \left(\frac{\partial \mathbf{t}}{\partial x_1}, \frac{\partial \mathbf{t}}{\partial x_2}, \dots, \frac{\partial \mathbf{t}}{\partial x_n}\right) = \begin{pmatrix} \frac{\partial t_1}{\partial x_1} & \cdots & \frac{\partial t_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial t_n}{\partial x_1} & \cdots & \frac{\partial t_n}{\partial x_n} \end{pmatrix}.$$

The matrices are square Jacobian matrices, whose $(i, j)^{\text{th}}$ entry is

$$\mathbf{J}_{ij} = \frac{\partial x_i}{\partial t_j} \; or \; \frac{\partial t_i}{\partial x_j}.$$

The inverse function theorem states that the matrix inverse of the Jacobian matrix of an invertible function is the Jacobian matrix of the inverse function.