*Completion of Classical Mechanics and Electrodynamics with Three Temporal Dimensions* by RA Gillmann - 2023-08-10

Introduction

Let us begin with the following propositions: The change in position of a body is characterized by (1) translatory motion with its *extent* measured by length and duration, and (2) rotational motion with its change in *direction* in three dimensions. An immediate inference is that a change in position includes (a) length with direction in three dimensions and (b) duration with direction in three dimensions. The former (a) is commonplace but the latter (b) has been ignored. The purpose of this paper is to begin to rectify the neglect of duration in three dimensions.

The notion that there could be three dimensions of time has been explored as a theoretical possibility without understanding what it could mean. Its actual existence has been dismissed as an “impossibility”. The apparent hiddenness of the “extra dimensions” of time has been a problem for those who propose them. Here an actual, observable three-dimensional space of duration is explored for the first time.

Although the concept of duration in three dimensions is clear enough, lack of familiarity with it makes it seem more difficult than it needs to be. Appendix A provides some historical background. Appendix B provides some examples.

The paper proceeds thusly: First is given the frame of reference required to include three dimensions of motion. Second

In the first part the duality of length and time (duration) is shown, which leads to the second part in which the sufficiency of the Galilean transformation (GT) and its dual is shown. [Lorentz transformation is sufficient in a different way.]

The motion of a body is observed to be three-dimensional. Two measures of the extent of motion are used: its length (or distance) and duration (or time). Let us posit two propositions about the motion of a body: (1) the length and duration of a motion are necessary and sufficient to measure its extent of motion, and (2) the three dimensions of motion entails that length and duration each have three dimensions.

Here a three-dimensional space with metric of length and duration is introduced. Others have investigated a ‘three-dimensional time’ that is quite unlike the simple approach given here [refs].

Frame of reference system

Consider an idealized apparatus [i.e., with negligible friction, etc.] with two adjacent rigid rods linked to each other and moving alongside each other [cf. a monorail] in uniform motion at a fixed rate called the *clock rate*. Call this device a *clock-rod*, i.e., a rod and a linear clock. The rod at rest relative to the observer is called the *length rod*. The adjacent rod is called the *duration rod* as depicted in Figure 2.

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Figure 1. Clock-rod

Let many identical copies of a clock-rod be made with the same rate of motion. Let the rods be arranged in a cubic lattice and extended as much as needed so that in principle they can measure any length or duration. A three-dimensional lattice of such clock-rods made as dense as needed serves as a frame of reference for both length and duration in any direction. Call it a *frame of reference system*. Such a frame of reference system is the physical correlate to a rectilinear coordinate system for measuring lengths and durations in three dimensions of motion.

The line between each length rod and duration rod is a coordinate line. The arrangement of these make up a rectilinear coordinate system for length and duration.

Let the rods be given marks at regular intervals by rolling a wheel on them at uniform velocity, making a mark or marks every time it turns, so that the marks represent both length and duration. For each rod define a mapping to the real numbers: *x* for the length rod and *z* for the duration rod.

Three mutually orthogonal clock-rods describe the three dimensions of motion. Designation of origins for length and duration space makes them vector spaces. The three dimensions of length are called here *length space*. The three dimensions of duration are called here *duration space*.

The position of an *event* is an ordered pair of vectors [**x**, **z**], a length space vector **L** in length space and a duration space vector **D** in the duration frame. The *location* of a point event in length space is its displacement from the origin, and the *chronation* of a point in duration space is its dischronment from the origin.

A diagram of a mathematical equation

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Figure 2. Standard configuration of coordinate axes based on rod-clocks

Events are represented by an ordered pair of Euclidean vectors [**x**, **z**] of their location and chronation. Every point on a length rod adjacent to a point on the duration rod of a clock-rod represents a position with length rod location *x* and duration rod chronation *z*. Positions of two events in length and duration space of two events add:

The directed length *L* between two events [*x*1, *z*] and [*x*2, *z*] is the difference between *x*1 and *x*2 on the length rod relative to an arbitrary *z* on the duration rod:

which equals a length Δ*x.* The directed duration *D* between two events [*x*, *z*1] and [*x*, *z*2] is the difference between *z*1 and *z*2 on the duration rod relative to an arbitrary *x* on the length rod:

which equals a duration Δ*z*. Events are equal if and only if their length and duration vectors are equal:

Scalar multiplication of an event is defined as:

The inner products, norms, and metrics of events are then defined as:

The induced metrics for event space are the Euclidean metrics for the length space, called *distance,* *s*, and the Euclidean metric for duration space, *t*, called *distime*:

Thus event space is a metric space.

There are two vector spaces: ℝ3 plus a one dimensional event order: length space with time and duration space with distance. It is convenient for the origins of the length frame and the duration frame to be adjacent at [**0**,**0**].

By symmetry length and duration are duals: a mapping from length to duration maps a length rod to its duration rod. The inverse maps a duration rod to its length rod. The mapping is linear, one-to-one, and onto, and thus an isomorphism. [A linear transformation *T* : *V* → *W* is called an isomorphism if *T* is one-to-one and onto.]

Motion

A *particle* *motion* (or a body whose position is represented by a point) is composed of a continuous series of events [**x**i, **z**i]. One body in motion [**x**i, **z**i] relative to another body in motion [**x**j, **z**j] is the difference between them: [**x**i − **x**j, **z**i – **z**j].

*Displacement* is the vector difference between two length space vectors: **x**f – **x**i, the initial and final vectors. The vector difference between two duration vectors, **z**n – **z**0, the initial and final vectors, which is called here *dischronment*. *Diseventment* is an ordered pair of the displacement and dischronment: [**x**f – **x**i, **z**f – **z**i].

Length space carries a projection of duration space onto a one-dimensional parameter called *time*. Duration space carries a projection of length space onto a one-dimensional parameter called *distance*.

If the ordered pair represents a length space vector **x** given the duration space vector **z**, then the duration space vector is independent and the length space vector dependent, and the event is in the *time domain*. If the ordered pair represents a duration space vector **z** given the length space vector **x**, then the length space vector is independent, and the event is in the *distance domain*.

Events are ordered in one of two ways: (*a*) events in length space are ordered by an independent one-dimensional duration parameter *t*, called *time t*, so that events with the same time [from a common point in time] are in equivalence classes of *simultaneous* events; and (*b*) events in duration space are ordered by an independent one-dimensional length parameter, called *distance* *s*, such that events with the same distance [from a common point] are in equivalence classes of *equidistant* events.

Note regarding experiments in general: whether an experimental variable is independent or dependent is determined by the experimental set-up and does not change because of mathematical manipulations. In other words, a functional argument is not necessarily an independent variable within an experiment.

Rates of motion

In the following, *t* is the independent time parameter, *s* is the independent distance parameter, **x** is the displacement, and **z** is the dischronment. The time rate of displacement is the *velocity* **v**:

|  |  |  |
| --- | --- | --- |
|  |  | (1) |

Another is called here the *lenticity* **w** which is the *dischronment* rate of motion [ref; inverse speed is ambiguous], defined as

|  |  |  |
| --- | --- | --- |
|  |  | (2) |

The inverse of lenticity is a velocity-like quantity **u**, called here the *harmonic velocity* [ref]:

|  |  |  |
| --- | --- | --- |
|  |  | (3) |

because the addition of two harmonic velocities is by harmonic addition [ref]:

|  |  |  |
| --- | --- | --- |
|  |  | (4) |

The velocity of light is a harmonic velocity since it is the harmonic mean of reflected distances. [ref on light]

There is an analogous harmonic pace defined as the inverse of velocity, which is a lenticity-like quantity **y**, called here the *harmonic lenticity* since it adds harmonically [ref]:

|  |  |  |
| --- | --- | --- |
|  |  | (5) |

Because of the symmetry of length and duration, velocity and lenticity are duals.

The time rate of velocity is the acceleration **a**, and the distance rate of lenticity is the vector **b**, here called the *relentment*:

|  |  |  |
| --- | --- | --- |
|  |  | (8) |

Because of the symmetry of length and duration, acceleration and relentment are also dual to one another. For motion that is a (scalar or vector-valued) function of both a length vector and a duration vector such as ϕ(**x**, **z**), see below.

The clock rate functions as a conversion of length and duration:

|  |  |  |
| --- | --- | --- |
|  |  | (9) |

Note that *s* and *t* convert since they are both independent variables, whereas **x** and **z** convert because they are both dependent variables. The clock rate can equal the harmonic mean speed of light and so be the conversion factor from independent duration to independent length without a one-way light postulate. [see Taylor & Wheeler 1.2 and 1.4]

Define a mapping between length space and duration that exchanges length vectors for their corresponding duration vectors and exchanges time with corresponding distance parameters. Such a map is linear, one-to-one and onto. Thus length space with time is isomorphic with duration space with distance: they are duals.

Dual Euclidean transformations

Events are subject to two *Euclidean transformations* [also known as *rigid transformations*], which is an isometry that preserves the event metrics between every pair of coordinates in [**x**; **z**] and [**x′**; **z′**] in which the latter is moving at velocity **v** in length space and lenticity **w** in duration space [ref].

These comprise the two parts of the completed Galilei transformation. One is the length part, with the independent scalar time the same for all observers:

|  |  |  |
| --- | --- | --- |
|  |  | (10) |
|  |  | (11) |

The other is the duration part, with the independent scalar indistance not requiring transformation:

|  |  |  |
| --- | --- | --- |
|  |  | (12) |
|  |  | (13) |

The first Euclidean transformation applies to the length frame in which **x** and **x′** are displacements relative to two observers, **v** is the relative velocity of the primed to the unprimed length frame, and *t* is the independent duration parameter. Since time is an independent parameter, it is the same for all observers.

The second Euclidean transformation applies to the duration frame in which **z** and **z′** are dischronments relative to the two observers, **w** is the relative lenticity of the primed to the unprimed length frame, and *s* is the independent length parameter called here indistance. Since indistance is an independent parameter, it is the same for all observers.

The length frame transformations form a group known as the Galilei group *G* [ref]. Because of duality the duration frame transformations form an isometric group.

Motion represented by a length vector function of **x**(*t*) uses only the first part of the transformation. Motion represented by a duration vector function of **z**(*s*) uses only the second part of the transformation. Motion represented by a vector function of both length and duration vectors **ψ**(**x**(*t*), **z**(*s*)) includes both parts of the transformation.

Symmetries

There are four symmetries: (1) translation symmetry in length space: Equation (10a), (2) translation symmetry in duration space: Equation (10b), (3) directional symmetry (isotropy) in length space, and (4) directional symmetry (isotropy) in duration space.

The directional symmetry is easily expressed in spherical coordinates. For length space:

|  |  |  |
| --- | --- | --- |
|  |  | (14) |
|  |  | (11) |

For duration space:

|  |  |  |
| --- | --- | --- |
|  |  | (15) |
|  |  | (13) |

Derivation of the Wave Equation

*Electric Field*

The electric and magnetic fields are functions of length and duration in three dimensions each: **E** = **E**(**x**, **z**) and **B** = **B**(**x**, **z**). Accordingly, Faraday’s and Ampère-Maxwell’s laws are completed as follows [cf. “Derivation of the Wave Equation in Time” etc. [*here*](https://em.geosci.xyz/content/maxwell1_fundamentals/appendix/wave_eq_derivation.html)]:

and

*Note*: to recover the original laws set ∂*t* = (∂*z*12 + ∂*z*22 + ∂*z*32)½. There are three constitutive relations:

(*A*3)

The derivation proceeds as follows: First take the length curl of Faraday’s completed law shown in equation (A1):

Substitute the constitutive relation (*A*3) into equation (A6) to get the following expression in terms of only the fields **E** and **H**:

Assuming the physical properties are homogeneous throughout the length-duration domain, *μ*, *ϵ*, and *σ* can be moved out front of the derivative terms. This simplifies (*A*7):

If we further assume that we can take the first and second derivatives of **E**, we can either take the length space derivatives first or the duration space derivatives first. Equation (A8) can then be written as:

This expression is now solely in terms of **∇x** × **E** and **∇x** × **H**. Next use equations (*A*2), (*A*4), and (*A*5) to generate an equation with only **E**:

Then simplify the left hand side of (*A*9) by using the vector identity:

Recalling that **∇x · E** is zero in a homogenous space, the vector identity simply becomes

Substitute that into (*A*10) to get the following:

This is the wave equation for the electric field in length-duration domain. If the charge is zero, then σ = 0, and the wave equation with *με* = *c*−2 is

*Magnetic Field*

To derive the wave equation for **H**, we repeat the above derivation but start by taking the length curl of Ampere’s Law completed, shown in equation (*A*2):

The constitutive relations can be substituted into equation (*A*13) to get the following expressions in terms of only **E** and **H**:

We simplify (*A*14) just as we did before for the electric field.

We can assume that we can take the first and second derivatives of **E** and **H** and can either take the spatial derivatives or duration derivatives first. Equation (*A*15) can then also be written as:

These expressions are now in terms of **∇x** × **E** and **∇x** × **H**. Thus, we can use equation (*A*1) to generate an equation with only **H**. Then again use the vector identity (*A*10) and the fact that **∇x · H** is zero in a homogenous space to simplify the vector identity, which is then substituted into the wave equation:

Equation (*A*17) is then the wave equation for the magnetic field in the length-duration domain. If the current is zero, then *σ* = 0, and the wave equation with *εμ* = *c*−2 is

Thus the completed Maxwell equations show electromagnetism is a wave.

Invariance of the Wave equation

The wave function (**x**(*t*), **z**(*s*)) is defined in terms of three dimensions each of length and duration (see above):

|  |  |  |
| --- | --- | --- |
|  |  | (11) |

with displacement **x**(*t*) and dischronment **z**(*s*). The left hand side of the equation is dependent on length with time, and the right hand side is dependent on duration with indistance, so both length and duration parts of the transformation are needed. Each part of the wave equation is Euclidean invariant [cf. Rohlf].

figure

Consider the standard configuration for relativity in which motion is parallel to the *x1-z1* plane. The dual Euclidean transformations are given by (10). The *x*2, x3, z2, and z3 coordinates are directly seen to be invariant. What remains is the one-dimensional wave equation:

Because

the length derivative of the length frame transformation is:

so that

Similarly because

the duration derivative of the length frame transformation is:

so that

Thus the wave equation in the moving frame is:

The result is that the wave equation is form invariant under the Euclidean transformations in event space.

Light clock revisited

A light clock is a thought experiment in which a light beam in a vacuum reflects back and forth between two parallel mirrors, a distance L apart (see figure below). When the light beam returns to the first mirror, one unit of time passes (“the clock ticks”). [ref Feynman vol 1, section 15-4; Gilbert Newton Lewis and Richard Chace Tolman in "The Principle of Relativity, and Non-Newtonian Mechanics" in *Proceedings of the American Academy of Arts and Sciences*, 1909, 44: 709–726, see Figure 1 and explanation on page 714; Light Clocks and the Clock Hypothesis, Samuel C. Fletcher, *Foundations of Physics* volume 43, pages 1369–1383 (2013); A light clock satisfying the clock hypothesis of special relativity, Joseph West, 2007 *European Journal of Physics*, Volume 28, Number 4].

Since the distance between the mirrors is set by the experimenter, the independent quantity is distance, and duration is a dependent quantity. This is key to understanding the experiment properly. The rate measured is the mean round-trip pace, which may be presented as a speed that is the harmonic mean round-trip speed.

Consider three cases as illustrated below:

A picture containing text, clock, gauge

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Figure 4. Light clock cases

Consider a beam of light in a vacuum reflected back, as in a “light clock” [ref Feynman vol 1, section 15-4; Gilbert Newton Lewis and Richard Chace Tolman in "The Principle of Relativity, and Non-Newtonian Mechanics" in *Proceedings of the American Academy of Arts and Sciences*, 1909, 44: 709–726, see Figure 1 and explanation on page 714; Light Clocks and the Clock Hypothesis, Samuel C. Fletcher, *Foundations of Physics* volume 43, pages 1369–1383 (2013); A light clock satisfying the clock hypothesis of special relativity, Joseph West, 2007 *European Journal of Physics*, Volume 28, Number 4].

Case 1 on the left shows a light clock at rest, with a light beam reflecting longitudinally back and forth between two mirrors. In this frame the round trip longitudinal distance between the two mirrors is 2L, and the time of one round trip is T = 2*Lk*, where *k* is the mean pace of light *in vacuo*. The mean pace is

which may be presented as the harmonic mean speed of light, c̄.

Case 2 shows an observer moving with lenticity w|| longitudinally to the light clock. Apply the Galilean duration transformation: t′ = t – w||x, where x is the longitudinal axis. Since the distance L is independent, it does not change from observer to observer. Let the pace of light in the first leg be *k*1 and the second leg *k*2. The duration of the first leg is then (Lk1 − w||L), and the duration of the second leg is (Lk2 + w||L). The duration of one round trip is

T = (Lk1 − w||L) + (Lk2 + w||L) = L(*k*1 + *k*2) = 2*kL* = 2*L*/*c̄*.

The mean pace is again *k*, shown as the harmonic mean speed of light, c.

Case 3 shows an observer moving with lenticity w⟂ transversely to the light clock. In this case there are two components of duration: longitudinal and transverse. These components are independent of one another since they are in different dimensions. The transverse component has no effect on the longitudinal component. The longitudinal motion is the same as the stationary case above: the total distance is 2L, the total duration is 2L/c, and the mean pace is again *k*, or the harmonic mean speed of light, c̄.

This result can be generalized: any reflected velocity is invariant under the Galilean transformation.

This shows an elementary mistake made in the Michelson-Morley experiment [ref] which compared the longitudinal and transverse cases [ref]:

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*Figure 5. Michelson-Morley apparatus*

They mistakenly calculated the mean round-trip duration (in the notation here) as

|  |  |  |
| --- | --- | --- |
|  |  | (12a) |

Since the independent variable is the distance traversed, with time as a dependent variable, so the experiment is in the distance (indistance) domain in which velocities are added harmonically, which results in

|  |  |  |
| --- | --- | --- |
|  |  | (12b) |

The longitudinal and transverse mean speeds should have been expected to be equal, which is what the Michelson-Morley experiment found but didn’t expect. Since the one-way velocity of light is not measurable according to the conventionality thesis, one could represent the forward and return speeds of light as *c*1 and *c*2, and the result would be the same. The initial direction has no effect on the result, which is also consistent with Michelson-Morley. Since the velocity *v* drops out, the total time is independent of the velocity. [cf. Kennedy-Thorndike, ref: Robertson] [Note: the author takes no position on the existence of a luminiferous ether or an emission theory of light.]

Dynamics

Because of the inverse relation of velocity and lenticity, *mass*, as the scalar constant associated with each body relative to length, relative to duration is naturally associated with the inverse of mass, called here *vass* [in*v*erse m*ass*]. [Similarly, the electric charge has a natural inverse, *varge* (in*v*erse ch*arge*).]

Since mass is the ratio of the measurand to the reference mass, the dual is the ratio of the reference mass to the measurand, which is the inverse mass, called here the *vass*. The dual of force is release.

The product of the body’s vass times its lenticity is called here the *fulmentum*. As force is the time rate of change of momentum, define *release* as the distance rate of change of fulmentum.

As gravitation is the attraction of mass, levitation is the attraction of vass: the less the mass, the greater the levitational attraction. In other words, a small mass is gravitationally attracted to a large mass, but a large vass is levitationally attracted to a small vass. Since the length and duration frames move in opposite directions, gravitational force and levitational release move in opposite directions.

Newton showed that force is proportional to mass, and since vass is the inverse of mass, so release is inversely proportional to mass and directly proportional to vass. Newton showed that force is also proportional to acceleration, so release is proportional to relentment: *R = nb*.

Duals of Newton’s laws of motion then are:

Dual First Law: Every body perseveres in a state of rest or of uniform motion in a straight line unless it is compelled to change that state by a net release.

Dual Second Law: The relentment of a body depends on the body’s vass and the amount of release applied.

Dual Third Law: If one body exerts a release on another body, then the second body exerts an equal and opposite release on the first body.

Note that an inertial or a facilial frame of reference can be defined as bodies for which the first law applies, which are two ways of referring to the same thing.

Newton showed that gravitational force is proportional to mass and inversely proportional to the distance squared. As vass is the inverse of mass, so levitational release is inversely proportional to mass and directly proportional to vass. Similarly, the levitational release is proportional to the distance squared with the direction opposite that of gravitational force: *R = L* *r*2/*mM*.

Symmetry and conservation

Length space rotation is symmetric for 3D length, and duration space rotation is symmetric for 3D duration. As 3D length translation symmetry implies conservation of linear momentum, so 3D duration translation implies conservation of linear fulmentum. As 3D length rotation symmetry implies conservation of angular momentum, so 3D duration translation symmetry implies conservation of angular fulmentum.

As 1D duration translation symmetrically orders events and implies energy conservation, so 1D length translations orders events and implies lethargy conservation. Length space and duration space are dual in any number of dimensions.

Conclusion

There are two ways to add velocities because there are two possible independent quantities, which leads to a frame system for classical physics. Two Euclidean transformations for length and duration make the wave equation classically invariant. Reflected light is classically invariant for all inertial observers. Length and duration are symmetric.

Since the Euclidean transformation has been shown here to be sufficient to explain the wave equation and the Michelson-Morely null result, some of the justification for the Lorentz transformation is removed. More research will be needed to determine if other justifications can be explained by the duality of length and duration.

Appendix A

Background

Consider how uniformly accelerated motion was described by Galileo:

If a moveable descends from rest in uniformly accelerated motion, the spaces run through in any times whatever are to each other as the duplicate ratio of their times; that is are as the squares of those times. (Galileo, *Two New Sciences*, “On Naturally Accelerated Motion,” tr. by Stillman Drake, p.166).

This is illustrated by Galileo’s explanation of a descending body [from page 221 – image from earlier translation]:

Diagram, engineering drawing

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Figure 1a – Galileo's Semi-Parabola Figure 1b – Vertical uniform motion

In *Figure 1a* a body moves uniformly from right to left on the horizontal line *abcde*. At *b* it begins to descend and follows the curve *bifh*, which Galileo shows is a semi-parabola. In his explanation he states:

Accordingly, we see that while the body moves from *b* to *c* with uniform speed, it also falls perpendicularly through the distance *ci*, and at the end of the time-interval *bc* finds itself at the point *i*.

Notice the shift of language: “the body moves from b to c” [i.e., a length], then “the time-interval bc”. Galileo uses a length to measure a time interval since the horizontal motion is uniform. That is, a body in uniform motion is observed as either an increasing length or duration.

Galileo shows that the vertical component of lengths as a semi-parabolic function of time intervals, represented by horizontal lines with increasing gaps. Given independent times, the ratio of two lengths of uniformly accelerated motion (*r*1, *r*2) are proportional to the ratio of the square of the corresponding times (*t*1, *t*2) of uniform motion:

*r*1 : *r*2 = *t*12 : *t*22,

for distances *ri* given times *ti* (*i* = 1, 2).

But *Figure 1b* shows Galileo could have equally well taken vertical lengths of uniform motion as the independent quantity and measured the times of descent relative to them. The vertical lengths would be independent values and the corresponding times would *not* be the elapsed time of a falling body but those of a uniform vertical motion measuring the extent of descent.

Given independent lengths of uniform motion, the corresponding times of descent are semi-parabolic for the same reason that lengths of descent are semi-parabolic given independent times of uniform motion. The ratio of two times of uniformly accelerated motion (*t*3, *t*4) are proportional to the ratio of the square of their corresponding lengths of uniform motion (*r*3, *r*4):

*t*3 : *t*4 = *r*32 : *r*42,

for spaces *ri* given times *ti* (*i* = 3, 4). To do this would require a frame of reference that incorporates the interchangeability of independent lengths and durations.

Galileo at first attempted to show that the time of descent was as the square of the distance but rejected it and argued against it. Others including Descartes argued for it but were not able to demonstrate it. In what follows the symmetry of the two positions is shown.

[cf. Galileo’s p.205 definition (times) and postulate (heights) for the experiment described – refs]

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