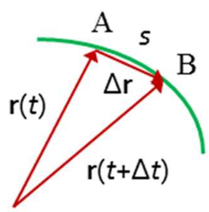
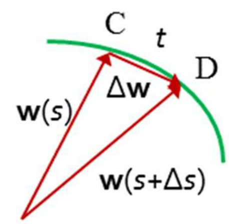
Parallel Differential Geometry of Curves

Based on the differential geometry part of the book *Shape Interrogation for Computer Aided Design and Manufacturing* by Nicholas M. Patrikalakis and Takashi Maekawa of MIT.

Let a three-dimensional *space curve* be expressed in parametric form as $x = x(t)$; $y = y(t)$; $z = z(t)$; where the coordinates of the point (x, y, z) of the curve are expressed as functions of a parameter t (time) within a closed interval $t_1 \le t \le t_2$. The functions $x(t)$, $y(t)$, and $z(t)$ are assumed to be continuous with a sufficient number of continuous derivatives.	Let a three-dimensional *time curve* be expressed in parametric form as $X = X(s)$; $Y = Y(s)$; $Z = Z(s)$; where the coordinates of the point (X, Y, Z) of the curve are expressed as functions of a parameter s (length) within a closed interval $s_1 \le s \le s_2$. The functions $X(s)$, $Y(s)$, and $Z(s)$ are assumed to be continuous with a sufficient number of continuous derivatives.
In vector notation the parametric curve can be specified by a vector-valued function $\mathbf{r} = \mathbf{r}(t)$, where $\mathbf{r}$ represents the position vector (i.e., $\mathbf{r}(t) = (x(t), y(t), z(t))$).	In vector notation the parametric curve can be specified by a vector-valued function $\mathbf{w} = \mathbf{w}(s)$, where $\mathbf{w}$ represents the position vector (i.e., $\mathbf{w}(s) = (X(s), Y(s), Z(s))$).
 Displacement $\Delta\mathbf{r}$ connecting points A and B on parametric curve $\mathbf{r}(t)$.	 Distimement $\Delta\mathbf{w}$ connecting points C and D on parametric curve $\mathbf{w}(s)$.
Consider a segment (displacement) of a parametric curve $\mathbf{r} = \mathbf{r}(t)$ between two points $P(\mathbf{r}(t))$ and $Q(\mathbf{r}(t+\Delta t))$ as shown in the figure above. As point B approaches A or in other words $\Delta t \to 0$, the length s becomes the differential *arc length* of the curve: $ds = \left\|\frac{d\mathbf{r}}{dt}\right\| dt = \|\mathbf{r}^t\| \, dt = \sqrt{\mathbf{r}^t \cdot \mathbf{r}^t} \, dt,$ where the superscript t denotes differentiation with respect to the *arc time parameter, t*.	Consider a segment (distimement) of a parametric curve $\mathbf{w} = \mathbf{w}(s)$ between two points $C(\mathbf{w}(s))$ and $D(\mathbf{w}(s+\Delta s))$ as shown in the figure above. As point D approaches C or in other words $\Delta s \to 0$, the time t becomes the differential *arc time* of the curve: $dt = \left\|\frac{d\mathbf{w}}{ds}\right\| ds = \|\mathbf{w}^s\| \, ds = \sqrt{\mathbf{w}^s \cdot \mathbf{w}^s} \, ds,$ where the superscript s denotes differentiation with respect to the *arc length parameter, s*.
So $\mathbf{r}^t = \frac{d\mathbf{r}}{dt}$, which is called the *tangent vector* at point A.	So $\mathbf{w}^s = \frac{d\mathbf{w}}{ds}$, which is called the *tangent vector* at point C.
The chain rule shows $\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt}.$	The chain rule shows $\frac{d\mathbf{w}}{ds} = \frac{d\mathbf{w}}{dt}\frac{dt}{ds}.$
For the magnitude of the tangent vector, $\text{since } \left\|\frac{d\mathbf{r}}{ds}\right\| = 1, \text{then } \|\mathbf{r}^t\| = \left\|\frac{d\mathbf{r}}{dt}\right\| = \left\|\frac{ds}{dt}\right\| = v,$ which is the speed at A.	For the magnitude of the tangent vector, $\text{since } \left\|\frac{d\mathbf{w}}{dt}\right\| = 1, \text{then } \|\mathbf{w}^s\| = \left\|\frac{d\mathbf{w}}{ds}\right\| = \left\|\frac{dt}{ds}\right\| = u,$ which is the *pace* at C.
The arc length, s, of a segment of the curve between points $\mathbf{r}(t_0)$ and $\mathbf{r}(t)$ is as follows: $s(t) = \int ds = \int \sqrt{\mathbf{r}^t \cdot \mathbf{r}^t} \; dt.$	The arc time, t, of a segment of the curve between points $\mathbf{w}(s_0)$ and $\mathbf{w}(s)$ is as follows: $t(s) = \int dt = \int \sqrt{\mathbf{w}^s \cdot \mathbf{w}^s} \; ds.$

The *unit tangent vector* is $$\mathbf{T}_s = \frac{\mathbf{r}^t}{\lvert\mathbf{r}^t\rvert} = \frac{d\mathbf{r}}{dt} \div \frac{ds}{dt} = \frac{d\mathbf{r}}{ds} \equiv \mathbf{r}^s,$$ where the superscript s denotes differentiation with respect to the *arc length parameter*, s.	The *unit tangent vector* is $$\mathbf{T}_t = \frac{\mathbf{w}^s}{\lvert\mathbf{w}^s\rvert} = \frac{d\mathbf{w}}{ds} \div \frac{dt}{ds} = \frac{d\mathbf{w}}{dt} \equiv \mathbf{w}^t,$$ where the superscript t denotes differentiation with respect to the *arc time parameter*, t.
So $\lvert\mathbf{T}_s\rvert = \lvert\mathbf{r}^s\rvert = \left\lvert\frac{d\mathbf{r}}{ds}\right\rvert = 1.$	So $\lvert\mathbf{T}_t\rvert = \lvert\mathbf{w}^t\rvert = \left\lvert\frac{d\mathbf{w}}{dt}\right\rvert = 1.$
If $\mathbf{r}(s)$ is an arc length parametrized curve, then $\mathbf{r}^s$ is a unit vector, and hence $\mathbf{r}^s \bullet \mathbf{r}^s = 1$. Differentiating this relation, we obtain $\mathbf{r}^s \bullet \mathbf{r}^{ss} = \mathbf{T}_s \bullet \mathbf{T}_s^s = 0$, which states that $\mathbf{r}^{ss}$ is orthogonal to the tangent vector, provided it is not a null vector. The *unit normal vector* is $$\mathbf{N}_s = \frac{\mathbf{r}^{ss}}{\lvert\mathbf{r}^{ss}\rvert} = \frac{\mathbf{T}_s^s}{\lvert\mathbf{T}_s^s\rvert},$$ which has the direction and sense of $\mathbf{r}^{ss}(s)$ is called the *unit principal normal vector* at s.	If $\mathbf{w}(t)$ is an arc length parametrized curve, then $\mathbf{w}^t(t)$ is a unit vector, and hence $\mathbf{w}^t \bullet \mathbf{w}^t = 1$. Differentiating this relation, we obtain $\mathbf{w}^t \bullet \mathbf{w}^{tt} = \mathbf{T}_t \bullet \mathbf{T}_t^t = 0$, which states that $\mathbf{w}^{tt}$ is orthogonal to the tangent vector, provided it is not a null vector. The unit vector $$\mathbf{N}_t = \frac{\mathbf{w}^{tt}}{\lvert\mathbf{w}^{tt}\rvert} = \frac{\mathbf{T}_t^t}{\lvert\mathbf{T}_t^t\rvert},$$ which has the direction and sense of $\mathbf{w}^{tt}(t)$ is called the *unit principal normal vector* at t.
The plane determined by the unit tangent and normal vectors $\mathbf{T}_s(s)$ and $\mathbf{N}_s(s)$ is called the *osculating plane* at s.	The plane determined by the unit tangent and normal vectors $\mathbf{T}_t(t)$ and $\mathbf{N}_t(t)$ is called the *osculating plane* at t.
The curvature is $$\kappa_s \equiv 1/\rho = \lvert\mathbf{r}^{ss}(s)\rvert = \lvert\mathbf{T}_s^s(s)\rvert,$$ and its reciprocal ρ is called the *radius of curvature* at s. It follows that $$\mathbf{r}^{ss} = \mathbf{T}_s^s = \kappa_s\, \mathbf{N}_s.$$	The curvature is $$\kappa_t \equiv 1/\rho = \lvert\mathbf{w}^{tt}(t)\rvert = \lvert\mathbf{T}_t^t(t)\rvert,$$ and its reciprocal ρ is called the *radius of curvature* at t. It follows that $$\mathbf{w}^{tt} = \mathbf{T}_t^t = \kappa_t\, \mathbf{N}_t.$$
The vector $\mathbf{k}_s = \mathbf{r}^{ss} = \mathbf{T}_s^s$ is called the *curvature vector*, and measures the rate of change of the tangent along the curve. By definition κ_s is nonnegative, thus the sense of the normal vector is the same as that of $\mathbf{r}^{ss}(s)$. For a three-dimensional curve, the curvature is $$\kappa_s = \lvert\mathbf{r}^t \times \mathbf{r}^{tt}\rvert \,/\, \lvert\, \mathbf{r}^t \rvert^3.$$	The vector $\mathbf{k}_t = \mathbf{w}^{tt} = \mathbf{T}_t^t$ is called the *curvature vector*, and measures the rate of change of the tangent along the curve. By definition κ is nonnegative, thus the sense of the normal vector is the same as that of $\mathbf{w}^{tt}(t)$. For a three-dimensional curve, the curvature is $$\kappa_t = \lvert\mathbf{w}^s \times \mathbf{w}^{ss}\rvert \,/\, \lvert\, \mathbf{w}^s \rvert^3.$$
Here are some useful formulae of the derivatives of the space curve, $\mathbf{r}(s)$ with respect to the arc time, t.	Here are some useful formulae of the derivatives of the time curve, $\mathbf{w}(t)$ with respect to the arc length, s.
$v = s^t = \mathrm{d}s/\mathrm{d}t = \lvert\, \mathbf{r}^t \rvert = (\mathbf{r}^t \bullet \mathbf{r}^t)^{1/2} = 1/\lvert\, \mathbf{w}^s \rvert = 1/(\mathbf{w}^s \bullet \mathbf{w}^s)^{1/2},$	$u = t^s = \mathrm{d}t/\mathrm{d}s = 1/\lvert\, \mathbf{r}^t \rvert = 1/(\mathbf{r}^t \bullet \mathbf{r}^t)^{1/2} = \lvert\, \mathbf{w}^s \rvert = (\mathbf{w}^s \bullet \mathbf{w}^s)^{1/2},$
$a = s^{tt} = \mathrm{d}s^t/\mathrm{d}t = (\mathbf{r}^t \bullet \mathbf{r}^{tt}) \,/\, (\mathbf{r}^t \bullet \mathbf{r}^t)^{1/2}$ $= -\,(\mathbf{w}^s \bullet \mathbf{w}^{ss}) \,/\, (\mathbf{w}^s \bullet \mathbf{w}^s)^{4/2},$	$b = t^{ss} = \mathrm{d}t^s/\mathrm{d}s = -\,(\mathbf{r}^t \bullet \mathbf{r}^{tt}) \,/\, (\mathbf{r}^t \bullet \mathbf{r}^t)^{4/2}$ $= (\mathbf{w}^s \bullet \mathbf{w}^{ss}) \,/\, (\mathbf{w}^s \bullet \mathbf{w}^s)^{1/2},$
$s^{ttt} = \mathrm{d}s^{tt}/\mathrm{d}t$ $= [(\mathbf{r}^t \bullet \mathbf{r}^t)(\mathbf{r}^t \bullet \mathbf{r}^{ttt} + \mathbf{r}^{tt} \bullet \mathbf{r}^{tt}) - (\mathbf{r}^t \bullet \mathbf{r}^{tt})^2] \,/\, (\mathbf{r}^t \bullet \mathbf{r}^t)^{3/2}$ $= -\,[(\mathbf{w}^s \bullet \mathbf{w}^s)(\mathbf{w}^s \bullet \mathbf{w}^{sss} + \mathbf{w}^{ss} \bullet \mathbf{w}^{ss}) - 4(\mathbf{w}^s \bullet \mathbf{w}^{ss})^2]$ $/\, (\mathbf{w}^s \bullet \mathbf{w}^s)^{7/2}.$	$t^{sss} = \mathrm{d}t^{ss}/\mathrm{d}s$ $= -\,[(\mathbf{r}^t \bullet \mathbf{r}^t)(\mathbf{r}^t \bullet \mathbf{r}^{ttt} + \mathbf{r}^{tt} \bullet \mathbf{r}^{tt}) - 4(\mathbf{r}^t \bullet \mathbf{r}^{tt})^2] \,/\, (\mathbf{r}^t \bullet \mathbf{r}^t)^{7/2}$ $= [(\mathbf{w}^s \bullet \mathbf{w}^s)(\mathbf{w}^s \bullet \mathbf{w}^{sss} + \mathbf{w}^{ss} \bullet \mathbf{w}^{ss}) - (\mathbf{w}^s \bullet \mathbf{w}^{ss})^2] \,/\, (\mathbf{w}^s \bullet \mathbf{w}^s)^{3/2}.$