**Henri Poincaré: a decisive contribution to Relativity**

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Translated from the French

https://www.lajauneetlarouge.com/henri-poincare-une-contribution-decisive-a-la-relativite/

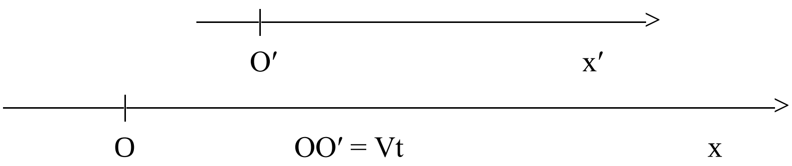
Revised from the translation http://www.annales.org/archives/x/Relativity.doc

## Appendix

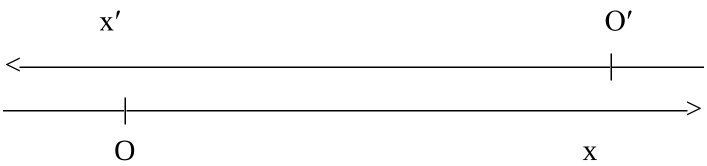
## The Lorentz transformation

It is essential to note that the Lorentz transformation is a direct consequence of the principle of relativity and does not require the invariance of the speed of light.

Let us look for this transformation along two axes Ox and O′x′ moving along each other with the constant relative velocity V.




In order to obtain perfect symmetry between the two frames of reference, let us put O′x′ in the other direction.




Homogeneity will lead to a linear transformation, and if we choose $t = t' = 0$ when the two origins O and O′ cross each other, the transformations $(x, t) \rightarrow (x', t')$ and $(x', t') \rightarrow (x, t)$ will be given as follows with eight appropriate constants from A to D′:

(4) $x' = Ax + Bt$ ; $t' = Cx + Dt$

$x = A'x' + B't'$ ; $t = C'x' + D't'$

The Principle of Relativity and symmetry lead to:

(5) $A = A'$; $B = B'$; $C = C'$; $D = D'$

Furthermore, we have at the origin O′: $x' = 0$ and $x = Vt$, hence from $x' = Ax + Bt$ we deduce $0 = AV + B$, and from $x = Ax' + Bt'$; $t = Cx' + Dt'$ we deduce $B = DV$, and thus $D = -A$.

Finally, consistency requires:

(6) $x = Ax' + Bt' = A(Ax + Bt) + B(Cx + Dt) = (D^2 + CDV)x$

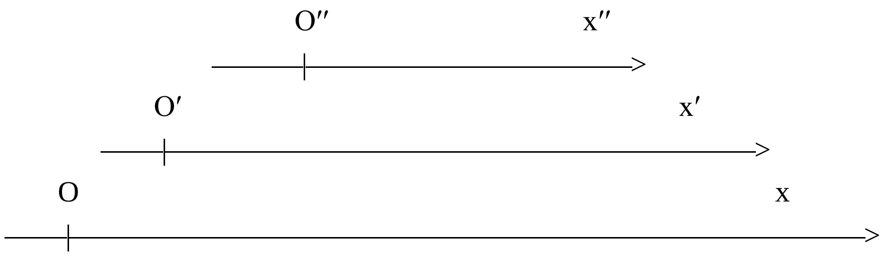
$t = Cx' + Dt' = C(Ax + Bt) + D(Cx + Dt) = (D^2 + CDV)t$

hence we need $D^2 + CDV = 1$, that is $C = (1 - D^2) / DV$, and the transformation $(x, t) \rightarrow (x', t')$ becomes:

(7) $x' = -Dx + DVt$ ; $t' = [(1 - D^2) / DV]\, x + Dt$

The only remaining unknown, D, is a function of the velocity and can be obtained through the comparison of several velocities.

Let us reverse again the axis O′x′ and let us consider the three axes Ox, O′x′, O″x″ in the same direction.




$OO' = Vt$; $O'O'' = V't'$; $OO'' = V''t$

With the opposite sign for x′ the relation (7) becomes:

(8) $x' = Dx - DVt)$; $t' = [(1 - D^2) / DV]x + Dt$

and thus similarly, with D′ for V′ and D″ for V″ [this new D′ is independent from that of (4)-(5) that is not used after (5)].

(9) $x'' = D'x' - D'V't')$; $t'' = [(1 - D'^2) / D'V'] x' + D't'$

(10) $x'' = D''x - D''V''t)$; $t'' = [(1 - D''^2) / D''V''] x + D''t$

The elimination of x′ and t′ in (8) and (9) gives another expression of (10):

(11) $x'' = \{DD' + [D'V'(D^2 - 1) / DV]\}x - DD'(V + V')t$

$t'' = \{[(D - DD'^2) / D'V'] + [(D' - D^2D') / DV]\}x + \{DD' + [DV(D'^2 - 1) / D'V']\}t$

The identification of (10) and (11) leads to the four following equalities:

(12) $D'' = DD' + [D'V'(D^2 - 1) / DV]$

(13) $D''V'' = DD' (V + V')$

(14) $(1 - D''^2) / D''V'' = [(D - DD'^2) / D'V'] + [(D' - D^2D') / DV]$

(15) $D'' = DD' + [DV(D'^2 - 1) / D'V']$

Hence with (12) and (15):

(16) $D'' - DD' = D'V'(D^2 - 1) / DV = DV( D'^2 - 1) / D'V'$

The last equality of (16) allows the definition of the quantity K by:

(17) $K = D^2 V^2 / (D^2 - 1) = D'^2 V'^2 (D'^2 - 1)$

The quantity K has the same value for two arbitrary velocities (and their corresponding D), hence it is a constant for all velocities. On the other hand, the case V= 0 gives x = x′ and t = t′, so D = 1 in (8) hence we must choose the positive solution of (17):

(18) $$D = 1 / \sqrt{1 - (V^2 / K)}$$

With (8) we thus reach the transformation (x, t) → (x′, t′), and Poincaré extends it without difficulty to the general transformation (x, y, z, t) → (x′, y′, z′, t′):

(19) $x' = (x - Vt) / \sqrt{1 - ( V^2 / K)}$ ; $y' = y$; $z' = z$; $t' = [t - (Vx / K)] / \sqrt{1 - ( V^2 / K)}$

It remains to determine the constant K that gives the Galilean transformation (1) if $K = \infty$ and the ordinary Lorentz transformation if $K = c^2$. [These two transformations are very near to each other if the ratio V/c is small.]

The constant K cannot be negative (it would become possible to go back in time) and its square root appears as an unsurpassable limit velocity. This latter point is confirmed by the expression $\sqrt{1 - ( V^2 / K)}$ and also by the composition of velocities that is given by (12) and (13):

(20) $V'' = (V + V') / [1 - (VV'/ K)]$

that is, with $\sqrt{K} = k$:

(21) $$\frac{k - V''}{k + V''} = \frac{(k - V)}{(k + V)} \cdot \frac{(k - V')}{(k + V')}$$

hence $\{ |V| < k$ , and $|V'| < k\}$ implies $|V''| < k$ .

Very naturally Poincaré and Lorentz have chosen $K = c^2$, which agrees with the invariance of the velocity of light and with the conservation of Maxwell′s equations in all inertial frames of reference. We can notice however that, if necessary, it remains possible that the constant K is very slightly larger than $c^2$. The photons would then have a very small but nonzero mass and their velocity, the velocity of light, would be a very slightly increasing function of their energy and would tend towards $\sqrt{K}$ as their energy increased indefinitely.

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(1) $x_1 = x - Vt$ : constant velocity V of the second frame of reference with respect to the first. $y_1 = y$ ; $z_1 = z$ ; $t_1 = t$ : absolute time.