Lorentz-like transformation derivation for time-distance domain

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What follows are derivations of the Lorentz transformation for the time domain and a Lorentz-like transformation for the distance domain.

Time Domain

Consider two inertial frames of reference O and O', assuming O to be at rest while O' is moving with velocity v with respect to O in the positive x-direction. The origins of O and O' initially coincide with each other. A light signal is emitted from the common origin and travels as a spherical wave front. Consider a point P on a spherical wavefront at a length r_x and r_x' from the origins of O and O' respectively.

For round-trip motion, the mean round-trip speed of light, c, is the same in all inertial frames, so for the point P with length vector $\mathbf{r} = (r_x, 0, 0)$ and

$$r_x = ct, \text{ and } r_x' = ct'.$$

Consider the standard Galilean transformation with a factor γ, which is to be determined and may depend on β, where $\beta = v/c$:

$$r_x' = \gamma(r_x - vt) = \gamma(r_x - \beta ct) = \gamma(r_x - \beta ct) = \gamma r_x(1 - \beta).$$

The inverse transformation is the same except that the sign of β is reversed:

$$r_x = \gamma(r_x' + vt') = \gamma(r_x' + \beta ct') = \gamma(r_x' + \beta ct') = \gamma r_x'(1 + \beta).$$

Multiply these two equations together to get

$$r_x r_x' = \gamma^2\, r_x r_x'(1 - \beta^2).$$

Divide out $r_x r_x'$ to get

$$\gamma^2 = 1/(1 - \beta^2),$$

$$\text{or } \gamma = 1/\sqrt{(1 - \beta^2)}.$$

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Consider what happens if this approach is applied to the independent time t times the constant c.

$$ct' = r_x' = \gamma(\mathrm{c}t - \beta r_x) = \gamma(r_x - \beta r_x) = \gamma r_x(1 - \beta).$$

The inverse transformation is the same except that the sign of β is reversed:

$$ct = r_x = \gamma(\mathrm{c}t' + \beta r_x') = \gamma(r_x' + \beta r_x') = \gamma r_x'(1 + \beta).$$

These are exactly the same equations as before, so the result will be the same.

Distance Domain

Consider two inertial frames of reference O and O', assuming O to be at rest while O' is moving with rapidity υ with respect to O in the positive x-direction. The origins of O and O' initially coincide with each other. A light signal is emitted from the common origin and travels as a spherical wave front. Consider a point Q on a spherical wavefront at a duration τ_x and τ_x' from the origins of O and O' respectively.

For round-trip motion, the mean round-trip pace of light, κ, is the same in all inertial frames, so for the point Q with duration vector $\boldsymbol{\tau} = (\tau_x, 0, 0)$ and

$$\tau_x = \kappa s, \text{ and } \tau_x' = \kappa s'.$$

Consider the dual standard Galilean transformation with a factor λ, which is to be determined and may depend on $\alpha = \upsilon/\kappa$ (cf. c/v)

$$\tau_x' = \lambda(\tau_x - \upsilon s) = \lambda(\tau_x - \alpha\kappa s) = \lambda(\tau_x - \alpha\tau_x) = \lambda\tau_x(1 - \alpha).$$

The inverse transformation is the same except that the sign of β is reversed:

$$\tau_x = \lambda(\tau_x' + \upsilon s') = \lambda(\tau_x' + \alpha\kappa s') = \lambda(\tau_x' + \alpha\tau_x') = \lambda\tau_x'(1 + \alpha).$$

Multiply these two equations to get

$$\tau_x\tau_x' = \lambda^2\tau_x\tau_x'(1 - \alpha^2).$$

Divide out $\tau_x\tau_x'$ to get

$$\lambda^2 = 1/(1 - \alpha^2), \text{ or}$$

$$\lambda = 1/\sqrt{(1 - \alpha^2)}.$$

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Consider what happens if this approach is applied to the independent distance s times the constant κ.

$$\kappa s' = \tau_x' = \lambda(\kappa s - \alpha\tau_x) = \lambda(\tau_x - \alpha\tau_x) = \lambda\tau_x(1 - \alpha).$$

The inverse transformation is the same except that the sign of α is reversed:

$$\kappa s = \tau_x = \lambda(\kappa s' + \alpha\tau_x') = \lambda(\tau_x' + \alpha\tau_x') = \lambda\tau_x'(1 + \alpha).$$

These are exactly the same equations as before, so the result will be the same.