Duality of Three-Dimensional Length and Duration in Newtonian Mechanics and Electrodynamics

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This paper presents a frame of reference system in which length and duration are a duality in three dimensions. With this dual Newtonian mechanics and the Galilean transformations are completed. Maxwellian electrodynamics and the wave equation are completed and shown to be Galilean invariant. A re-examination of the light clock and the Michelson-Morley experiment confirms the Galilean invariance of the speed of light for all inertial observers. Throughout the paper the significance of the difference between independent and dependent variables is highlighted.

Keywords: Multidimensional time, duality of space and time, Newtonian mechanics, Galilean transformations, electrodynamics, Maxwell's equations, wave equation, speed of light, Michelson-Morley experiment

INTRODUCTION

Uniform Motion

Let us begin with some historical background. Galileo gave two definitions of uniform motion. One definition was:

> Equal or uniform motion I understand to be that of which the parts run through by the moveable in any equal times whatever are equal to one another. [1, p. 148]

In other words, for uniform motion if two equal time intervals are given, the distances traversed will be equal as well. Galileo also defined uniform motion another way:

> If a moveable equably carried [latum] with the same speed passes through two spaces, the times of motion will be to one another as the spaces passed through. [1, p. 149]

That is, for uniform motion if two distances ("spaces") are given, the times of motion will be proportional. Both definitions are implicit in the Eudoxan proportion:

$$x_m : x_n :: t_m : t_n. \tag{1}$$

with corresponding distance intervals, x_m and x_n, and time intervals, t_m and t_n. Uniform motion can also be expressed as a proportion of rates:

$$x_m : t_m :: x_n : t_n. \tag{2}$$

But such proportions are ambiguous: which one is measured with respect to the other, time or distance? In other words, which variable is independent, and which is dependent?

Galileo's dual definitions result from interchanging the independent and dependent variables. These lead to different kinematic domains depending on which variable is independent.

Uniformly Accelerated Motion

Next consider how uniformly accelerated motion was described by Galileo:

> If a moveable descends from rest in uniformly accelerated motion, the spaces run through in any times whatever are to each other as the duplicate ratio of their times; that is are as the squares of those times. [1, p. 166]

This is illustrated by Galileo's explanation of a body in free fall [1, p. 221]. In *Figure 1* a body moves uniformly from right to left on the horizontal line *ABCDE*. At *B* it begins to descend and follows the curve *BIFH*, which Galileo shows is a semi-parabola. In his explanation he states:

> Accordingly, we see that while the body moves from *B* to *C* with uniform speed, it also falls perpendicularly through the distance *CI*, and at the end of the time-interval *BC* finds itself at the point *I*. [1, p. 221]

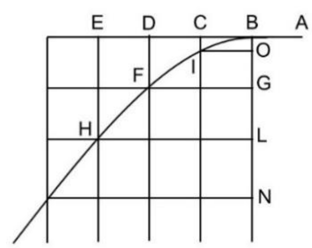

Figure 1. Galileo's semi-parabola

Notice the shift of language: *the body moves from B to C* [i.e., a distance], then *the time-interval BC*. Galileo uses a *distance* to measure a *time* interval. That is, the body is observed as either an increasing distance or increasing time. The horizontal uniform motion is independent of the vertical free fall and acts as a reference motion of either distance or time.

At first Galileo considered free fall with respect to the distance of descent but after a false attempt, he related free fall to the time and velocity of descent [2] [3] [4]. Descartes and others defended the first approach but were unable to state it correctly.[1] Below we show free fall with respect to the *elapsed distance*.

This paper proceeds as follows: In the first part the duality of length and duration is shown, which leads to a dual Newtonian mechanics. In the second part the Galilean transformations[2] are completed, and the invariance of the completed equations of Maxwell and the completed wave

[1] See Chap. 9 of [2], Chap. 3 of [3], and Chap. 3 of [4].

[2] Or Galilei transformations.

equation are shown. In the third part the light clock and Michelson-Morley experiment are re-examined.

Others have considered a duality between "space and time", e.g., [5], or the possibility of three dimensions of time or six dimensional spacetime in the context of relativity theory, at least since the 1970s, e.g., [6] [7] [8] [9]. But they all differ from the approach developed here, which is based on Newtonian mechanics and Maxwellian electrodynamics.

DUALITY DEFINED

Frame of Reference System

Consider these two facts about the nature of a body in motion: (1) there are two measures of the extent of motion: length and duration[3]; (2) the space of motion is three-dimensional. These two measures in three dimensions determine two three-dimensional vector spaces, one with a length metric and the other with a duration metric.

Consider an idealized apparatus with two rigid rods, connected but moveable, with one rod moving adjacent to the other in uniform motion at a fixed rate called the *elapse rate*.[4] Call this device a *clock-rod* because it functions as a linear rod and a linear clock.

Call one rod the *length rod* and the adjacent rod the *duration rod*. Each rod moves relative to the other at the elapse rate but in the opposite direction. Figure 2 depicts the duration rod moving relative to the length rod:

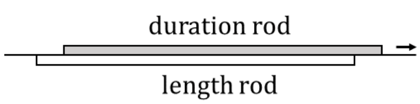

Figure 2. Clock-rod

Let many identical copies of the first clock-rod be made. Let the rods be arranged into a cubic lattice so that in principle they can measure any length or duration in any direction.[5]

Let the line between each pair of length and duration rods represent a coordinate line. Three mutually orthogonal clock-rods represent axes of a rectilinear coordinate system for length and duration. The length rods in three dimensions comprise the *length frame*, and the duration rods in three dimensions comprise the *duration frame*.

A system of clock-rods in three dimensions is a *frame of reference system*. The coordinate lines between each length rod and duration rod is modelled by a rectilinear coordinate system for measuring lengths and durations in three dimensions of motion.

The three dimensional real vector space $\mathbb{R}^3$ of length is called *length space*. The three dimensional real vector space $\mathbb{R}^3$ of duration is called *duration space*. Length space and duration space have the same coordinate lines since their length rods and duration rods are adjacent. Directions in length space and duration space are collinear and opposite.

A vector of length space is called the *displacement*, and its magnitude is called the *traversal distance*. A vector of duration space is called here the *dischronment*[6], and its magnitude is called the *traversal time*.

In *Figure 3* a body moves from A to A' relative to the length rod, and the length of motion is the difference between A and A' on the length rod scale. During this motion a mark on the duration rod moves from B to B' relative to the length rod. The time of motion is the difference between B (adjacent to A) and B' on the duration rod scale.

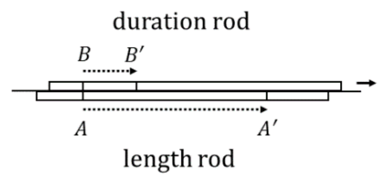

Figure 3. Length and time measurement

In *Figure 4* a body moves from A to A' relative to the duration rod, and the duration of motion is the difference between A and A' on the duration rod scale. During this motion a mark on the length rod moves from B to B' relative to the duration rod. The distance of motion is the difference between B (adjacent to A) and B' on the length rod scale.

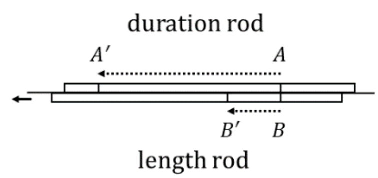

Figure 4. Duration and distance measurement

In this way the measures of length and duration are defined in terms of a reference uniform motion.

Motion

An event is represented by a point on a frame of reference system. The position of an event in length space is its *location* on the length rods. In duration space the position of an event on the duration rods is called here its *chronation*. Events are the same if their locations and chronations are equal.

The motion of a body is comprised of a continuous series of events, which are ordered by an independent variable,

[3] Duration could be called *time* but here time is restricted to its meaning as a scalar parameter.
[4] Cf. a monorail train.
[5] Other rods and arrangements are possible, e.g., straight and annular rods in a spherical lattice.
[6] I.e., *dis* (away) + *chron* (time) + *ment*.

which is a dynamic parameter that forms a chain of causality. Events in length space are ordered by the duration of an independent uniform motion, which is called *elapsed time*. Length space with elapsed time is called the *time domain*.

Events in duration space are ordered by the length of an independent uniform motion, which is called *elapsed distance*[7]. Duration space with elapsed distance is called the *distance domain*. Events for which both time and distance are independent variables are in the *time-distance domain*, which is comprised of the time domain and the distance domain.

An observer at rest relative to the length frame is in the time domain. An observer at rest relative to the duration frame is in the distance domain. An observer at rest relative to the duration frame is in motion relative to an observer at rest relative to the length frame, and *vice versa*.

Since elapsed time order and elapsed distance order are physically independent, there is no conversion between the time domain and the distance domain. The time domain and distance domain form a complete domain for motion.

By construction length and duration are symmetric. Bijective maps from length space to duration space and elapsed time to elapsed distance and *vice versa*, establish a complete duality between the time domain and the distance domain.

Rates of Motion

A rate of change is a ratio with an independent variable in the denominator. For rates of motion the independent variable is either elapsed time in the time domain or elapsed distance in the distance domain. In the following, t is elapsed time, s is elapsed distance, $\mathbf{x}$ is the displacement, and $\mathbf{z}$ is the dischronment.

In Table 1 velocity $\mathbf{v}$ is defined as the time rate of displacement, which is a length space vector. The distance rate of dischronment $\mathbf{w}$, called here *lenticity*[8], is a duration space vector.

Table 1. Rates of motion

Time Domain	*Distance Domain*
Velocity	Lenticity
$\mathbf{v} := \dfrac{d\mathbf{x}}{dt}$	$\mathbf{w} := \dfrac{d\mathbf{z}}{ds}$

Since velocity is in the time domain, velocity is a function of the independent variable *time*. Lenticity is in the distance domain and is a function of the independent variable *distance*. Faster motions are indicated by larger velocities and smaller lenticities.

Instantaneous speed is the magnitude of velocity, which over time is the time rate of arc length traversed such as the speed of a journey leg. The magnitude of lenticity, which over distance is the distance rate of arc duration traversed, is called *pace*[9].

Motion measured with respect to distance but presented as a speed is a distance rate speed, which is an inverse pace (*Table 2*). Motion measured with respect to time but presented as a pace is a time rate pace, which is an inverse speed. The speed of light measured by reflection is an example of distance rate speed.

Table 2. Inverse rates of motion

Distance Domain	*Time Domain*
Inverse Pace, Distance Rate Speed	Inverse Speed, Time Rate Pace
$u = w^{-1} = \left(\dfrac{dz}{ds}\right)^{-1}$	$v^{-1} = \left(\dfrac{dx}{dt}\right)^{-1}$

These inverses must be inverted in order to be added (or subtracted), then inverted again. This is called here *harmonic addition (or harmonic sum)*[10] [10] [11] defined as:

$$u_1 \boxplus u_2 := (u_1^{-1} + u_2^{-1})^{-1} \tag{3}$$

with zero replacing any division by zero. The average distance rate speed is their harmonic mean over a distance.[11]

Table 3. Rates of rates of motion

Time Domain	*Distance Domain*
Acceleration	Relentation
$\mathbf{a} := \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{x}}{dt^2}$	$\mathbf{b} := \dfrac{d\mathbf{w}}{ds} = \dfrac{d^2\mathbf{z}}{ds^2}$

The time rate of velocity is the acceleration $\mathbf{a}$, and the distance rate of lenticity is the vector $\mathbf{b}$, here called the *relentation*[12] (*Table 3*). Because of the duality of length and duration, acceleration and relentation are also dual to one another.

Other dual quantities can be defined by inverting and interchanging length and duration. Equations of motion in the time domain have dual equations in the distance domain and *vice versa*, as in the following.

DUALITY ELABORATED

[7] I.e., *distance* here is an independent length, e.g., the distance along a race course, which is not necessarily the Euclidean distance.
[8] I.e., slowness.
[9] This meaning of *pace* is from racing and traffic flow theory.

[10] In electrical engineering this operation is known as *parallel addition* or *reciprocal addition*.
[11] Cf. the space mean speed of traffic flow theory.
[12] Spanish, slowing.

Dual Newtonian Mechanics

Scalar properties of bodies measured on a ratio scale such as mass are dual with their inverse ratio. In *Table 4* mass m is dual to the *inverse mass*, which is shortened here to *vass* n. The mass-weighted velocity in the time domain is the momentum ($\mathbf{p}$), and the vass-weighted lenticity in the distance domain is called here the *levamentum*[13] ($\mathbf{q}$).

Table 4. Weighted rates of motion

Time Domain	*Distance Domain*
Momentum	Levamentum
$\mathbf{p} := m\dfrac{d\mathbf{v}}{dt}$	$\mathbf{q} := n\dfrac{d\mathbf{w}}{ds}$

The interaction of bodies (or a body and its environment) in the time domain is called *force* ($\mathbf{F}$). In the distance domain an interaction of bodies is called here *release*[14] ($\mathbf{R}$).

Newton's axioms or laws of motion in the time domain are given with their duals in the distance domain below [12] [13]:

> *First Time Domain Law:* A body remains at rest or in uniform motion unless acted on by a force.

> *First Distance Domain Law:* A body remains at rest or in uniform motion unless acted on by a release.

Newton's first law is a statement of the time domain law of inertia, which corresponds to the distance domain *law of facilia*[15]. A body in the time domain resists change, whereas in the distance domain a body remains in the same state unless there is an 'easier' motion. They amount to the same law, a law of inertia-facilia, which is the condition for an *inertial-facilial frame of reference system*.

There is no change of velocity or lenticity in an inertial-facilial frame of reference system. An inertial-facilial frame of reference system moving at a constant velocity or constant lenticity relative to an inertial-facial frame of reference system is also inertial-facilial. All of the frame of reference systems considered here are inertial-facilial.

> *Second Time Domain Law:* A body impacted by a force moves in such manner that the time rate of change of momentum equals the force.

> *Second Distance Domain Law:* A body impacted by a release moves in such manner that the distance rate of change of levamentum equals the release.

The second laws concern the cause of changes of motion, which are force in the time domain and release in the distance domain (*Table 5*).

Table 5. Cause of changes of motion

Time Domain	*Distance Domain*
Force	Release
$\mathbf{F} := \dfrac{d\mathbf{p}}{dt}$	$\mathbf{R} := \dfrac{d\mathbf{q}}{ds}$

> *Third Time Domain Law:* If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction.

> *Third Distance Domain Law:* If two bodies enable releases on each other, these releases are equal in magnitude and opposite in direction.

The third laws undergird the conservation of momentum and levamentum, about which see below.

Gravitation and Levitation Duality

Newton's law of gravitation in the time domain postulated a gravitational force $\mathbf{F}_{21}$ exerted on mass m_2 by mass m_1 as

$$\mathbf{F}_{21} = -G\frac{m_1 m_2}{r_{12}^2}\hat{\mathbf{r}}_{12}$$

where the negative sign indicates an attractive force toward m_2, the constant of proportionality is G, and $\hat{\mathbf{r}}_{12}$ is a unit vector directed from mass m_1 to mass m_2. [14]

The gravitational acceleration g for a body with mass m is then

$$g = -G\frac{m}{r^2}.$$

The dual to gravitational force in the distance domain is called here *levitational release* $\mathbf{R}_{21}$ enabled on vass n_2 by vass n_1 as

$$\mathbf{R}_{21} = -L\frac{n_1 n_2}{\rho_{12}^2}\hat{\boldsymbol{\rho}}_{12}$$

where the negative indicates an attractive release toward n_2, the constant of proportionality is L, and $\hat{\boldsymbol{\rho}}_{12}$ is a unit vector directed from n_1 to n_2.

The levitational relentation ℓ for a body with vass n is then

$$\ell = -L\frac{n}{\rho^2}.$$

In other words, gravitational force and levitational release are dual explanations of the same phenomenon.

Dual Semi-Parabola

The dual to Galileo's semi-parabola can now be stated. Figure 1 diagrams the semi-parabola in the time domain:

$$x = x_0 - gt^2$$

with traversal height x, initial location (height) x_0, acceleration of gravity g, and elapsed time t.

[13] Latin, alleviation.
[14] Cf. releasing a restraint.
[15] Latin, *easy*.

Let the experiment take place in duration space, with an independent distance. This distance is an *elapsed distance* rather than a traversal distance, and the corresponding time is not the elapsed time but the *traversal time* of a uniform vertical motion measuring the time of descent.

The semi-parabola in the distance domain is

$$z = z_0 - \ell s^2$$

with traversal duration z, initial chronation z_0, relentation of levity ℓ, and elapsed distance s.

Symmetry and Conservation

The length frame transformations form a group known as the Galilei group G. [15] Because of duality the duration frame transformations form an isometric group.

The symmetries are: (a) translation symmetry of elapsed time and elapsed distance, and (b) translation symmetry and isotropy of length space and duration space.

Length space rotation is symmetric (isotropic) for three-dimensional length, and duration space rotation is symmetric (isotropic) for three-dimensional duration. As three-dimensional length translation symmetry implies conservation of linear momentum, so three-dimensional duration translation implies conservation of linear levamentum.

As three-dimensional length rotation symmetry implies conservation of angular momentum, so three-dimensional duration translation symmetry implies conservation of angular levamentum. As elapsed time translations imply energy conservation, so elapsed distance translations imply the conservation of lethargy, the inverse of energy.

Clearly, this can be continued so that there is a dual for all of Newtonian mechanics.

Dual Galilean Transformations

The dual Galilean transformations are between two inertial-facilial frame systems in the time-distance domain. The time-distance domain is the time domain and the distance domain together. It consists of two three-dimensional vector spaces and two independent variables, time and distance.

Let the length-space axis $\mathbf{x_1}'$ be moving at velocity $\mathbf{v}$ relative to length-space axis $\mathbf{x_1}$ (*Figure 5*). Consider two observers at rest relative to coordinate systems K and K' with K' moving with velocity $\mathbf{v}$ along the x_1-x_1' axis relative to K.

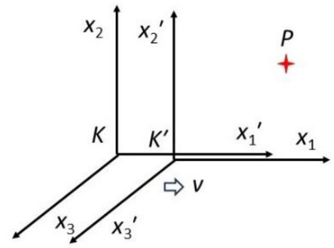

Figure 5. Two observers in the time domain

Let event P be observed at $\mathbf{x} = (x_1, 0, 0)$ with time t in K and $\mathbf{x'} = (x_1', 0, 0)$ with time t' in K'. The Galilean transformation in the time domain in this configuration is

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \mathbf{v}t \\ t' &= t \end{aligned} \tag{4}$$

The universality of elapsed time is a consequence of its independence in the time domain[16]. The independence of a time series in the time domain is the source of the impression that elapsed time "flows" of its own accord. [12, p. 408]

Let the duration-space axis $\mathbf{z_1}'$ be moving at lenticity $\mathbf{w}$ relative to duration-space axis $\mathbf{z_1}$ (*Figure 6*). Consider two observers at rest relative to coordinate systems S and S' with S' moving with lenticity $\mathbf{w}$ along the z_1-z_1' axis relative to S.

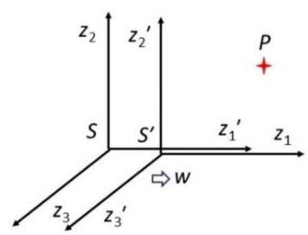

Figure 6. Two observers in the distance domain

Let event P be observed at $\mathbf{z} = (z_1, 0, 0)$ with distance s in S and $\mathbf{z'} = (z_1', 0, 0)$ with distance s' in S'. The Galilean transformation in the distance domain in this configuration is

$$\begin{aligned} \mathbf{z}' &= \mathbf{z} - \mathbf{w}s \\ s' &= s \end{aligned} \tag{5}$$

The universality of elapsed distance is a consequence of its independence in the distance domain.

A length vector function $\mathbf{x}(t)$ uses only the time domain transformation, Equation (4). A duration vector function $\mathbf{z}(s)$ uses only the distance domain transformation, Equation (5). A vector function of both length and duration vectors such as $\psi(\mathbf{x}(t), \mathbf{z}(s))$ includes both transformations.

Complete Wave Equation

Consider a wave function $\psi(\mathbf{x}, \mathbf{z})$ in the time-distance domain with displacement $\mathbf{x}$, dischronment $\mathbf{z}$, and constant

[16] I.e., Newton's first Scholium on absolute time and space is not relevant for his mechanics (p. 408ff [12]).

c. The displacement and dischronment are relatively independent.

The complete wave equation in the time-distance domain is defined with vector functions of displacement $\mathbf{x}$ and dischronment $\mathbf{z}$ as:

$$\left(\nabla_{\mathbf{x}} - \frac{1}{c^2}\nabla_{\mathbf{z}}^2\right)\psi(\mathbf{x},\mathbf{z}) = 0 \tag{6}$$

with constant of proportionality c^{-2}. The complete electromagnetic field is derived below as an example of a complete wave equation in the time-distance domain.

Complete Electric Field

The complete electric and magnetic fields are functions with both length and duration in three dimensions: $\mathbf{E} = \mathbf{E}(\mathbf{x}, \mathbf{z})$ and $\mathbf{B} = \mathbf{B}(\mathbf{x}, \mathbf{z})$, cf. [16, pp. 18-1 to 18-4]. Faraday's law and Ampère-Maxwell's law are completed as follows [17]:

$$\nabla_{\mathbf{x}} \times \mathbf{E} = -\left(\frac{\partial \mathrm{B}_1}{\partial \mathrm{z}_1}, \frac{\partial \mathrm{B}_2}{\partial \mathrm{z}_2}, \frac{\partial \mathrm{B}_3}{\partial \mathrm{z}_3}\right) = -\nabla_{\mathbf{z}}\mathbf{B} \tag{7}$$

and

$$c^2\nabla_{\mathbf{x}} \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \left(\frac{\partial E_1}{\partial z_1}, \frac{\partial E_2}{\partial z_2}, \frac{\partial E_3}{\partial z_3}\right) = \frac{\mathbf{j}}{\epsilon_0} + \nabla_{\mathbf{z}}\mathbf{E}. \tag{8}$$

In a vacuum the charges and currents equal zero, so Equation (8) becomes [16, pp. 20-7 to 20-14]:

$$c^2\nabla_{\mathbf{x}} \times \mathbf{B} = \nabla_{\mathbf{z}}\mathbf{E}. \tag{9}$$

The electromagnetic wave is derived as follows: first take the length curl of Faraday's complete law shown in equation (7):

$$\nabla_{\mathbf{x}} \times (\nabla_{\mathbf{x}} \times \mathbf{E}) = -\nabla_{\mathbf{z}}(\nabla_{\mathbf{x}} \times \mathbf{B}). \tag{10}$$

Now the curl of the curl of any vector can be written as the sum of two terms, one with the divergence and the other the Laplacian:

$$\nabla_{\mathbf{x}} \times (\nabla_{\mathbf{x}} \times \mathbf{E}) = \nabla_{\mathbf{x}}(\nabla_{\mathbf{x}} \cdot \mathbf{E}) - \nabla_{\mathbf{x}}^2\mathbf{E}.$$

Since in a vacuum the divergence of $\mathbf{E}$ is zero, only the Laplacian term remains. From Equation (8) in a vacuum the duration derivative of $c^2\nabla_{\mathbf{x}} \times \mathbf{B}$ is the second partial derivative of $\mathbf{E}$ with respect to $\mathbf{z}$:

$$c^2\nabla_{\mathbf{z}}(\nabla_{\mathbf{x}} \times \mathbf{B}) = \nabla_{\mathbf{z}}^2\mathbf{E}.$$

Equation (10) then becomes:

$$\nabla_{\mathbf{x}}^2 = \frac{1}{c^2}\nabla_{\mathbf{z}}^2\mathbf{E}, \tag{11}$$

which is the complete wave equation. Written out it reads:

$$\frac{\partial^2\mathbf{E}}{\partial x_1^2} + \frac{\partial^2\mathbf{E}}{\partial x_2^2} + \frac{\partial^2\mathbf{E}}{\partial x_3^2} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial z_1^2} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial z_2^2} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial z_3^2} = 0.$$

Thus the complete Maxwell equations show that electromagnetism is a wave.

Invariance of the Wave Equation

The wave function $\psi(\mathbf{x}, \mathbf{z})$ in the time-distance domain is defined above in Equation (6) with displacement $\mathbf{x}$, and dischronment $\mathbf{z}$, and constant c. Apply the dual Galilean transformations to show that the wave equation is invariant.

The left hand side of Equation (6) is dependent on length with elapsed time, and the right hand side is dependent on duration with elapsed distance, so both length and duration transformations are needed. Each part of the two-way wave equation is Galilean invariant. [18, pp. 104-105]

Consider a standard configuration (Figure 2) in which motion is parallel to the x_1-z_1 plane. The Galilean transformations are given by Equations (4) and (5). The x_2, x_3, z_2, and z_3 coordinates are directly seen to be invariant. What remains is the one-dimensional wave equation:

$$\frac{\partial^2\psi}{\partial x_1^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial z_1^2}.$$

Because

$$\frac{\partial x_1'}{\partial x_1} = 1 \text{ and } \frac{\partial z_1'}{\partial x_1} = 0,$$

the *length* derivative of the time domain transformation is:

$$\frac{\partial\psi}{\partial x_1} = \frac{\partial\psi}{\partial x_1'}\frac{\partial x_1'}{\partial x_1} + \frac{\partial\psi}{\partial z_1'}\frac{\partial z_1'}{\partial x_1} = \frac{\partial\psi}{\partial x_1'}$$

so that

$$\frac{\partial^2\psi}{\partial x_1^2} = \frac{\partial^2\psi}{\partial {x'}_1^2}$$

Similarly because

$$\frac{\partial x_1'}{\partial z_1} = 0 \text{ and } \frac{\partial z_1'}{\partial z_1} = 1,$$

the *duration* derivative of the distance domain transformation is:

$$\frac{\partial\psi}{\partial z_1} = \frac{\partial\psi}{\partial x_1'}\frac{\partial x_1'}{\partial z_1} + \frac{\partial\psi}{\partial z_1'}\frac{\partial z_1'}{\partial z_1} = \frac{\partial\psi}{\partial z_1'}$$

so that

$$\frac{\partial^2\psi}{\partial z_1^2} = \frac{\partial^2\psi}{\partial {z'}_1^2}$$

Thus the two-way wave equation in the moving frame is:

$$\frac{\partial^2\psi}{\partial \mathbf{x}'^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial \mathbf{z}'^2} \tag{12}$$

The result is that the wave equation is form invariant under the dual Galilean transformations in event space.

CONSEQUENCES OF DUALITY

Light Clock Re-examined

A *light clock* is a thought experiment in which a light beam in a vacuum reflects back and forth between two parallel mirrors, a distance L apart (see *Figure 4* below). When the light beam returns to the first mirror, one period of time passes. [19, pp. 15-9 to 15-11] [20] [21] [22]

Since light moves between mirrors whose distance is set by the experimenter, the independent variable is *elapsed distance*, and the dependent variable is duration. The rate of motion measured is the mean round-trip pace of light, which is inverted as the harmonic mean round-trip speed c in a vacuum. Although the one-way speed of light is not measured, it can be nominally set to c in the context of a round-trip.[17]

Consider three cases of a light clock in *Figure 7*.

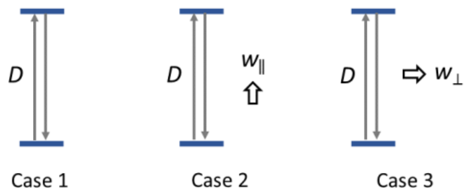

Figure 7. Light clock cases

The distance between the mirrors is independently set at D, and so is the same for all observers.

Case 1 shows a light clock at rest relative to the observer, with a light beam reflecting longitudinally back and forth between two mirrors. In this frame the round trip distance between the two mirrors is $2D$, and the period of one round trip is

$$T = \frac{2D}{c}. \tag{13}$$

Case 2 shows an observer moving with lenticity $w_{||}$ longitudinally to the light clock. Apply the Galilean distance domain transformation: $t' = t - w_{||}x$, where x is the longitudinal axis. Since the distance L is independent, i.e., is set by the experimenter, it does not change from observer to observer.

Nominally, the duration of the first leg is

$$T_1 = (D/c - Dw_{\parallel}),$$

and the duration of the second leg is

$$T_2 = (D/c + Dw_{\parallel}).$$

The period T of one round trip is:

$$T = (D/c - Dw_{\parallel}) + (D/c + Dw_{\parallel}) = \frac{2D}{c}. \tag{14}$$

Case 3 shows an observer moving with lenticity $w_{\perp}$ transversely to the light clock. In this case there are two components of duration: longitudinal and transverse, which are independent of one another since they are in different dimensions. Transverse motion has no effect on longitudinal motion, so the longitudinal motion is the same as the stationary case above: the total distance is $2D$, the period T of one round trip is again $2D/c$.

Michelson-Morley Experiment Re-examined

The Michelson-Morley experiment, [23] compared the longitudinal and transverse cases of reflected light, expecting to detect an ether wind (*Figure 8*).

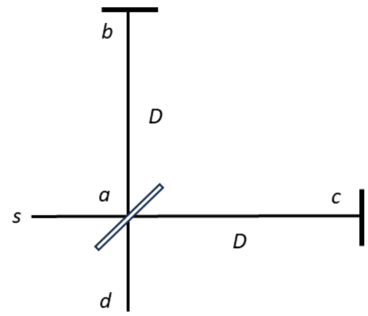

Figure 8. Michelson-Morley apparatus

They explain: "Let *sa* … be a ray of light which is partly reflected in *ab*, and partly transmitted in *ac*, being returned by the mirrors *b* and *c*, along *ba* and *ca*. [Then] *ba* is partly transmitted along *ad*, and *ca* is partly reflected along *ad*. If then the paths *ab* and *ac* are equal [distances D], the two rays interfere along *ad*."

By rotating the apparatus they expected to detect an ether wind parallel to *ba* or *ca*. They calculated the round-trip duration (in the notation here) as

$$T = \frac{D}{c - v} + \frac{D}{c + v}. \tag{15}$$

But since the independent variable is the distance traversed, duration is a *dependent* variable, and the experiment is in the distance domain. Speeds are distance rate speeds, which add by harmonic addition. So the denominators should be:

$$c \boxminus v = (c^{-1} - v^{-1})^{-1}$$

and

$$c \boxplus v = (c^{-1} + v^{-1})^{-1}.$$

The period of a round trip is then:

$$T = \frac{D}{c \boxminus v} + \frac{D}{c \boxplus v} = \frac{2D}{c}. \tag{16}$$

As with a light clock, the round-trip periods would be equal whether or not there was an ether wind, so their null result should have been expected.

CONCLUSION

Two themes have guided this paper: the significance of the distinction between independent and dependent variables

[17] Cf. the *Round-Trip Light Principle* [25]

and the duality of length and duration for bodies in motion.

An historical introduction found a duality of length and duration in Galileo's *Two New Sciences*. Led by this, the frame of reference was expanded into a frame of reference system in which length and duration are duals.

Then Newtonian mechanics, Maxwellian electrodynamics, and the Galilean transformations are completed with their duals. The Galilean invariance of the completed Maxwellian electrodynamics and the completed wave equation are shown. A re-examination of the light clock and the Michelson-Morley experiment confirms the Galilean invariance of the round-trip speed of light.

This removes two justifications for the Lorentz transformations [24] and raises the question whether it is needed. Additional research is required to determine to what extent the completed Galilean transformations are sufficient for physics.

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